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Vertex operators and composite supersymmetric S-functions

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Abstract. The S-function basis for the Fock space of vertex-operator constructions is considered, where states are associated with partitions $|\lambda\rangle$. We show that the matrix element $\langle\lambda|V_\alpha(z)|\mu\rangle$ is essentially given by the *composite* S-function $s_{\bar{\lambda};\mu}(\bar{z})$ where the argument $\bar{z} \equiv z^{-1}$ is replicated $|\alpha|$ times. Matrix elements of products of vertex operators $\langle\lambda|V_\alpha(z)V_{-\beta}(w)\cdots|\mu\rangle$ are given up to functions of $(z, w, \ldots)$ independent of λ and μ by the composite *supersymmetric* S-function $s_{\bar{\lambda};\mu}(\bar{z}, \ldots/\bar{w}, \ldots)$ where the $\bar{z}, \bar{w}, \ldots$ are replicated $|\alpha|, |\beta|, \ldots$ times, the supersymmetric arguments being those associated with negative roots.

As an application of these results, certain trace and product formulae in the vertex-operator constructions are verified using S-function identities for various infinite series. On the other hand, MacDonald identities formulated in terms of S-functions have a simple interpretation as vertex-operator matrix elements between appropriate states. In the case of level-one representations of Kac–Moody algebras (in particular $A_1^{(1)}$), the operator product expansion for the currents is verified for arbitrary matrix elements. Finally the Sugawara construction of the energy–momentum tensor for currents of arbitrary level is established using suitable regulated partial traces over the level-one 'reference' representation.

1. Introduction

Vertex operators were introduced originally in string theory, but have come to play a key role in many different contexts in mathematics and physics [1]. Besides their special use in string emission amplitudes, they are the general vehicle for the pervasive fermion–boson equivalence in two-dimensional field theory. More specifically, through the so-called Coulomb gas method, the vertex constructions are important in many aspects of conformal and superconformal field theory and applications [2]. On the mathematical side, the vertex construction provides a systematic way of obtaining concrete realizations of the basic (level-one) representations of affine Kac–Moody algebras [3] and superalgebras [4], which in turn are intimately related to conformal and superconformal symmetry; higher level constructions are also possible [5]. Further generalizations of the vertex operators describe mappings from the Heisenberg algebra into $gl(\infty)$ or $gl(\infty/\infty)$, and the formulation of hierarchies associated with such integrable nonlinear equations as the KdV and KP equations [6] has also relied on properties of such vertex constructions. Finally, certain classes of wavefunctions for multi-anyonic systems have been found to be intimately related to appropriate vertex-operator matrix elements [7].

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The theme of the present paper is that the theory of symmetric functions, or S-functions [8], provides an important tool for the analysis of vertex-operator constructions, which hitherto has not been fully exploited. The name derives from the seminal work of Schur on the theory of the symmetric group; the intimate relationship betwen the latter and the Weyl group of simple Lie algebras leads, in turn, to the application of S-functions to group characters and the representations of simple Lie algebras and superalgebras [9–11].

The link with S-functions in the case of vertex-operator constructions is not through group characters, or enumeration of irreducible representations, but rather via the structure of Fock space. Specifically, basis states of the form

$$(a_{-n})^{l_n}(a_{-n+1})^{l_{n-1}}\cdots(a_{-2})^{l_2}(a_{-1})^{l_1}|0\rangle \tag{1}$$

comprising negatively moded 'creation' operators with $l_n, l_{n-1}, \ldots, l_2, l_1 \geqslant 0$ acting on the 'vacuum' state $|0\rangle$, are naturally associated with partitions

$$\lambda = (n^{l_n}, (n-1)^{l_{n-1}}, \ldots, 2^{l_2}, 1^{l_1})$$

where $|\lambda| = l_n n + l_{n-1}(n-1) + \cdots + 2l_2 + 1l_1$ is the level (or occupation number). This suggests that kets can be succinctly labelled by all possible partitions $|\lambda\rangle$, so that if d is the level operator the 'q-dimension' would become, for example,

$$\sum_\lambda \langle\lambda|q^d|\lambda\rangle = \sum_\lambda q^{|\lambda|} = \prod_{i=1}^{\infty}(1-q^n)^{-1}$$

as expected, where the trace is over all Young frames λ.

This connection has been exploited to good effect and it can indeed be shown [12] that the entire calculus of S-functions and their generalizations has a natural meaning in terms of appropriate bases for Fock space. The vertex operators are introduced through their Laurent modes, and are related to partitions in the manner indicated in (1). The discussion has been extended to q-deformed vertex operators and associated algebras and q-deformed S-functions [13].

In the present paper we focus on the original (undeformed) field theoretical vertex operators $V_\alpha(z)$ (where the root α determines the conformal dimension $\frac{1}{2}\alpha^2$). In section 2 we introduce the S-function basis for Fock space and examine the matrix element $\langle\lambda|V_\alpha(z)|\mu\rangle$. Following earlier work [14], our main new result [15] is that it is given by the composite S-function $s_{\bar{\lambda};\mu}$ where the indeterminates forming its arguments are specialized to $\bar{z} \equiv 1/z$, replicated $|\alpha|$ times (the S-function can be formally defined for non-integral α in this case). A result such as this is perhaps not surprising: for example, it is to be expected from the levels of $|\lambda\rangle$ and $|\mu\rangle$, and confirmed by the degree of homogeneity of the S-function, that the matrix element is a function of $z^{(|\lambda|-|\mu|)}$, so that only certain Laurent coefficients of V_α are involved. However, the reformulation in terms of S-functions has far-reaching consequences for further manipulations of vertex operators. Thus as shown in section 2, products of vertex operators and matrix elements of the form $\langle\lambda|V_\alpha(z)V_{-\beta}(w)\cdots|\mu\rangle$ is given by the *supersymmetric* composite S-function $s_{\bar{\lambda};\mu}(\bar{z},\ldots/\bar{w},\ldots)$ with the arguments replicated $|\alpha|, |\beta|, \ldots$ times, the supersymmetric arguments being the ones associated with negative roots.

In section 3, as an application of these results, certain trace and product properties of the vertex operators are derived using various S-function identities [16, 17]. In particular, we compute the regulated trace $\sum_\lambda\langle\lambda|V_\alpha(z)V_{-\beta}(w)\cdots q^d|\mu\rangle$

by interpreting the resulting S-function sum as a supersymmetric MacDonald identity. Other MacDonald-type identities [16] can be interpreted in Fock space as resulting from different types of trace, or matrix elements between the ground state and certain 'reservoir' states, of products of vertex operators (see appendix 2).

In section 4 we reconsider the relationship between the vertex construction and representations of infinite-dimensional algebras. We first examine the singularities of matrix elements in the S-function basis of products of vertex operators when the points at which they are defined coalesce. Thus we verify the operator product expansions directly for the case of the level-one vacuum representation of $A_1^{(1)}$ ($\alpha = \pm\sqrt{2}$). In this case, a natural current operator $A(z)$ is introduced on the tensor product of an arbitrary representation with the basic representation as 'reference' representation. It is shown that, in an appropriate contour integral, the regulated partial trace of $A(z)A(w)$ over the 'reference' representation gives essentially the energy–momentum tensor, thus recovering the Sugawara construction from a trace formula in this framework.

The concluding remarks in section 5 provide an outlook for further extensions of the S-function calculus to applications in conformal field theory, statistical systems and their associated symmetry algebras.

2. Vertex operators in the Schur function basis

We work with the untwisted vertex operator,

$$V_\alpha(z) = \exp\left\{\alpha\sum_{n=1}^{\infty}\frac{z^n}{n}a_{-n}\right\}\exp\left\{-\alpha\sum_{n=1}^{\infty}\frac{z^{-n}}{n}a_n\right\}. \tag{2}$$

For convenience we will refer to the first exponential as $V_\alpha^+(z)$ and the second as $V_\alpha^-(z)$. The operators a_m satisfy the commutation relations of the Heisenberg algebra

$$[a_m, a_n] = m\delta_{m+n,0}. \tag{3}$$

For later reference we also introduce in the standard way a further (momentum) operator $a_0 \equiv p_0$ which commutes with the a_n and a_{-n} (so that the algebra for $m \in \mathbb{Z}$ remains that of (3)) and which is represented on the tensor product of the Fock space of the a_n with a Hilbert space associated with p_0 and its conjugate q_0, where $[q_0, p_0] = i$, with momentum states $|p_\lambda\rangle$ obtained by acting with $\mathrm{e}^{\mathrm{i}p_\lambda q_0}$ on the vacuum state $|0\rangle$. In section 4 we shall need the modified vertex operators including momenta,

$$U_\alpha(z) = z^{\alpha^2/2}V_\alpha(z)\mathrm{e}^{\mathrm{i}\alpha q_0}z^{\alpha p_0}. \tag{4}$$

A realization of the Heisenberg algebra, and hence of the vertex operators $V_\alpha(z)$ and $U_\alpha(z)$, in terms of symmetric functions is the concern of this section and the starting point of the applications to be discussed in the next two sections. First it is necessary to introduce some preliminary notation and definitions.

We use the conventions of MacDonald [8], as far as is possible. The *power sum* symmetric functions $p_n(x)$, $n = 1, 2, \ldots$ of indeterminates $x = (x_1, x_2, \ldots)$ are defined by the generating function

$$P(t) = \sum_{m=1}^{\infty} p_m(x)t^{m-1} = \frac{\mathrm{d}}{\mathrm{d}t}\log\prod_{i=1}^{\infty}(1 - x_i t)^{-1}. \tag{5}$$

For a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ define $p_\lambda = p_{\lambda_1} p_{\lambda_2} \ldots$. Then the $p_\lambda(x)$ form a basis for the symmetric functions of the x_i.

There is a wide variety of other integer and non-integer bases for the ring of symmetric functions [8] in addition to the power sums. Of particular concern are the so-called *elementary* symmetric functions $e_n(x)$ and the *complete* symmetric functions $h_n(x)$, which can be defined through their generating functions

$$E(t) = \sum_{m=0}^{\infty} e_m(x) t^m = \prod_i (1 + x_i t) \qquad H(t) = \sum_{m=0}^{\infty} h_m(x) t^m = \prod_i (1 - x_i t)^{-1}. \tag{6}$$

These are, in fact, both special cases of the the Schur or S-functions $s_\lambda(x)$ which, like the $p_\lambda(x)$, are labelled by arbitrary partitions λ and which possess determinantal expansions in terms of the $p_n(x)$, $e_n(x)$ and $h_n(x)$ [8]. It turns out that $h_n(x) \equiv s_{(n)}(x)$, and $e_n(x) \equiv s_{(1^n)}(x)$. In the case of the $p_n(x)$ the relationship to the Schur functions entails the Frobenius formula for the characters of the symmetric group,

$$p_n(x) = \sum_{|\rho|=n} \chi^{\rho}_{(n)} s_\rho(x). \tag{7}$$

More generally there exist *transition matrices* expressing the basis transformations between the various sets of symmetric functions. In section 4 we shall require formula (7) and write $\chi^{\rho}_{(n)} \equiv K^{-1}{}_{(n)\rho}$ where the Kostka matrix $K_{\lambda\mu}$ in general gives the S-functions in terms of yet another set, the monomial symmetric functions.

There is a natural inner product $\langle , \rangle$ under which $p_\lambda(x)$ form an orthogonal set. If one defines the operation $D(f)$ as the adjoint of multiplication by the function f, then [8] $D(p_n(x)) = n\partial/\partial p_n(x)$. In particular, $D(p_n(x))$ is a derivation on the space of symmetric functions of x. We thus have a realization of the Heisenberg algebra (3) in the space of symmetric functions by associating

$$a_{-n} \leftrightarrow p_n(x) \qquad \text{and} \qquad a_n \leftrightarrow D(p_n(x))$$

where n is positive; p_0 can be represented in the manner described earlier if required. From this point of view the link discussed above between Fock space number states and partitions is simply the selection of the $p_\lambda(x)$ as a preferred basis. However the Schur or S-functions themselves form another $\mathbb{Z}$ basis for the ring of symmetric functions, *orthonormal* with respect to the same inner product $\langle , \rangle$, and is a more natural basis to use for labelling states in Fock space for representing the vertex operators, as we shall see.

The action of the vertex operator in this basis is facilitated by describing the vertex operator itself as a linear combination of S-functions. The following S-function series will figure prominently in what follows:

$$J_q(x;z) \equiv \sum_\lambda q^{|\lambda|} s_\lambda(x) s_\lambda(z) = \prod_{i,j=1}^{\infty} (1 - q x_i z_j)^{-1} \tag{8}$$

$$I_q(x;z) \equiv \sum_\lambda (-q)^{|\lambda|} s_{\lambda'}(x) s_\lambda(z) = \prod_{i,j=1}^{\infty} (1 - q x_i z_j) \tag{9}$$

where λ' is the partition conjugate to λ, $|\lambda|$ is the length of the partition, and the sums run over all partitions. We use the generating function (5) for $p_n(x)$ and the identities (8) and (9) to write

$$V_\alpha^+(z) = \prod_{i\geqslant 1}(1-x_i z)^{-\alpha}$$
$$= \begin{cases} \left(\sum_\lambda s_\lambda(z)s_\lambda(x)\right) & \alpha > 0 \\ \left(\sum_\lambda (-1)^{|\lambda|} s_\lambda(z)s_{\lambda'}(x)\right) & \alpha < 0. \end{cases}$$

Here z is (formally, for α not a positive integer) such that $z_1 = z_2 = \cdots = z_\alpha = z$ and $z_{\alpha+1} = z_{\alpha+2} = \cdots = 0$, and $s_\lambda(z)$ is given by [8]

$$s_\lambda(z) = \binom{\alpha}{\lambda'} z^{|\lambda|} \tag{10}$$

where the generalized binomial coefficient is defined as

$$\binom{\alpha}{\lambda} = \prod_{n\in\lambda} \frac{\alpha - c(n)}{h(n)}$$

with $c(n)$ and $h(n)$ being, respectively, the content and hook length corresponding to node n of the partition λ. The same treatment is applied to $V_\alpha^-(z)$ resulting in the following realization of the vertex operator:

$$V_\alpha(z) = \begin{cases} \left\{\sum_\lambda s_\lambda(z)s_\lambda(x)\right\}\left\{\sum_\rho (-1)^{|\rho|} s_\rho(\bar{z}) D(s_{\rho'}(x))\right\} & \alpha > 0 \\ \left\{\sum_\lambda (-1)^{|\lambda|} s_\lambda(z)s_{\lambda'}(x)\right\}\left\{\sum_\rho s_\rho(\bar{z}) D(s_\rho(x))\right\} & \alpha < 0 \end{cases} \tag{11}$$

where we use the notation $\bar{z} \equiv 1/z$.

To calculate matrix elements we require the notion of S-function multiplication defined by $s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda$ with $c^\lambda_{\mu\nu}$ being the Littlewood–Richardson coefficients, and that of a skew S-function $s_{\lambda/\mu}$ defined by $s_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu\nu} s_\nu$. Using the orthonormality of the S-functions, the vertex-operator matrix element $(V_\alpha(z))_{\mu\nu}$, defined by

$$(V_\alpha(z))_{\mu\nu} \equiv \langle s_\mu(x), V_\alpha(z)s_\nu(x)\rangle$$

can be calculated to be

$$(V_\alpha(z))_{\mu\nu} = \begin{cases} (-1)^{|\nu|}\sum_\lambda (-1)^{|\lambda|} s_{\mu/\lambda}(z)s_{\nu'/\lambda'}(\bar{z}) & \alpha > 0 \\ (-1)^{|\mu|}\sum_\lambda (-1)^{|\lambda|} s_{\mu'/\lambda}(z)s_{\nu/\lambda'}(\bar{z}) & \alpha < 0. \end{cases}$$

We note that this can be expressed in terms of a composite S-function or 'universal character' introduced by King [11, 16] in the context of representations of Lie algebras and Lie superalgebras and defined as

$$s_{\bar{\nu};\mu}(x) = \sum_\xi (-1)^{|\xi|} s_{\nu/\xi}(\bar{x}) s_{\mu/\xi'}(x). \tag{12}$$

Two other concepts are of importance here: that of an S-function of a compound argument, defined by

$$s_\mu(x,y) = \sum_\lambda s_{\mu/\lambda}(x) s_\lambda(y)$$

and that of a supersymmetric S-function, defined by

$$s_\mu(x/y) = \sum_\lambda (-1)^{|\lambda|} s_{\mu/\lambda}(x) s_{\lambda'}(y).$$

Supersymmetric S-functions figure prominently in the study of representations of superalgebras, but were already known to Littlewood [9, 10, 16]. Composite supersymmetric S-functions can also be defined, as in (12) but with the argument x replaced by x/y. In terms of these S-functions we can write the vertex operator matrix element as

$$(V_\alpha(z))_{\mu\nu} = \begin{cases} (-1)^{|\nu|} s_{\bar\mu;\nu'}(\bar z) & \alpha > 0 \\ (-1)^{|\nu|} s_{\bar\mu;\nu'}(0/\bar z) & \alpha < 0. \end{cases} \tag{13}$$

Thus far, we have made the distinction between positive and negative α explicit. However, we can obtain a unified description if we adopt the notation $((z_1, z_2, \ldots, z_n))$ for the S-function argument which has all the z_is corresponding to all the positive (negative) α_is to the left (right) of the slash in $(\ldots/\ldots)$. Then, the matrix element (13) is given by

$$(V_\alpha(z))_{\mu\nu} = (-1)^{|\nu|} s_{\bar\mu;\nu'}((\bar z)).$$

The vertex operator (11) itself can now be written simply as

$$V_\alpha(z) = \left\{ \sum_\lambda s_\lambda((z)) s_\lambda(x) \right\} \left\{ \sum_\rho (-1)^{|\rho|} s_\rho((\bar z)) D(s_{\rho'}(x)) \right\}$$

regardless of the sign of α. We can also introduce the series

$$I_q((x_1, \ldots, x_n; y_1, \ldots, y_n)) \equiv \sum_\lambda (-q)^{|\lambda|} s_{\lambda'}((x_1, \ldots, x_n)) s_\lambda((y_1, \ldots, y_n))$$

being the generalization of (9) and calculated by repeated application of the identities

$$I_q(x, y; z, w) = I_q(x; z) I_q(x; w) I_q(y; z) I_q(y; w)$$
$$I_q(x/y; z/w) = I_q(x; z) I_q(y; w) J_q(x; w) J_q(y; z).$$

To evaluate matrix elements of a product of n vertex operators, we proceed as follows. In the case of $n = 2$ we have

$$(V_{\alpha_1}(z_1) V_{\alpha_2}(z_2))_{\mu\nu} = \sum_\lambda (-1)^{|\lambda|} s_{\bar\mu;\lambda'}((\bar z_1)) (-1)^{|\nu|} s_{\bar\lambda;\nu'}((\bar z_2)).$$

Define $B_{\mu,\nu}(\bar x/\bar y; \bar w/\bar z)$ to be

$$B_{\mu,\nu}(x/y; w/z) = \sum_\lambda (-1)^{|\lambda|} (-1)^{|\nu|} s_{\bar\mu;\lambda'}(\bar x/\bar y) s_{\bar\lambda;\nu'}(\bar w/\bar z).$$

Form the 'generating function' $\sum_{\mu\nu} B_{\mu\nu}(x/y;w/z)s_\mu(r)s_\nu(t)$, rewrite it as a series in another way and compare coefficients of $s_\mu(r)s_\nu(t)$ in the two expressions. We find that

$$
\begin{aligned}
\sum_{\mu\nu} & B_{\mu\nu}(x/y;w/z)s_\mu(r)s_\nu(t) \\
&= \sum_{\beta,\gamma,\lambda} s_\beta(r)s_\beta(x/y)(-1)^{|\gamma|}s_\gamma(t)s_{\gamma'}(\bar{w}/\bar{z}) \\
&\quad \times (-1)^{|\lambda|}s_\lambda(\bar{x}/\bar{y},r)s_\lambda(w,t/z) \\
&= J_1(r;x/y)I_1(t;\bar{w}/\bar{z})I_1(\bar{x}/\bar{y},r;w,t/z) \\
&= I_1(\bar{x}/\bar{y};w/z)J_1(r;t)J_1(r;x,w/y,z)I_1(t;\bar{x},\bar{w}/\bar{y},\bar{z}).
\end{aligned}
$$

where we have used results from appendix 1. The last three products can be written as

$$
\begin{aligned}
J_1 & (r;t)J_1(r;x,w/y,z)I_1(t;\bar{x},\bar{w}/\bar{y},\bar{z}) \\
&= \sum_{\beta,\gamma,\lambda} s_\beta(r)s_\beta(t)s_\gamma(r)s_\gamma(x,w/y,z)(-1)^{|\lambda|}s_\lambda(t)s_{\lambda'}(\bar{x},\bar{w},\bar{y},\bar{z}) \\
&= \sum_{\mu,\nu,\rho} s_\mu(r)s_\nu(t)(-1)^{|\nu|}s_{\bar{\mu};\nu'}(\bar{x},\bar{w}/\bar{y},\bar{z})
\end{aligned}
$$

giving the result

$$
B_{\mu,\nu}(x/y;w/z) = I_1(\bar{x}/\bar{y};w/z)(-1)^{|\nu|}s_{\bar{\mu};\nu'}(\bar{x}\bar{w}/\bar{y},\bar{z}). \tag{14}
$$

Applying (14), we obtain

$$
(V_{\alpha_1}(z_1)V_{\alpha_2}(z_2))_{\mu\nu} = I_1((z_2;\bar{z}_1))(-1)^{|\nu|}s_{\bar{\mu};\nu'}((\bar{z}_1,\bar{z}_2)).
$$

In fact, it can be proved by induction that the matrix element of a product of n vertex operators is given by

$$
\begin{aligned}
(V_{\alpha_1}(z_1)\cdots V_{\alpha_n}(z_n))_{\mu\nu} &= \prod_{1\leqslant i<j\leqslant n} I_1((z_j;\bar{z}_i))(-1)^{|\nu|}s_{\bar{\mu};\nu'}((\bar{z}_1,\bar{z}_2,\ldots,\bar{z}_n)) \\
&= \prod_{1\leqslant i<j\leqslant n}\left(1-\frac{z_j}{z_i}\right)^{\alpha_i\alpha_j}(-1)^{|\nu|}s_{\bar{\mu};\nu'}((\bar{z}_1,\bar{z}_2,\ldots,\bar{z}_n)).
\end{aligned} \tag{15}
$$

At the operator level, this says that

$$
\begin{aligned}
V_{\alpha_1}(z_1)\cdots V_{\alpha_n}(z_n) &= \prod_{i<j}\left(1-\frac{z_j}{z_i}\right)^{\alpha_i\alpha_j}\left\{\sum_\lambda s_\lambda((z_1,\ldots,z_n))s_\lambda(x)\right\} \\
&\quad \times \left\{\sum_\rho (-1)^{|\rho|}s_\rho((\bar{z}_1,\ldots,\bar{z}_n))D(s_{\rho'}(x))\right\}.
\end{aligned} \tag{16}
$$

Finally it should be remarked that arbitrary matrix elements of products of vertex operators can also be given in the number basis associated with the power-sum symmetric functions. The results [15] turn out to be expressed in terms of products of Charlier polynomials over the partition labels and the arguments of the vertex operators.

3. S-function identities and vertex-operator properties

Inherent in the S-function realization of the vertex operators is the possibility of bringing to bear the tools of S-functions in computing properties of the operators themselves. Here we transcribe various S-function identities and generating functions into the vertex-operator language. Some of the resulting vertex-operator identities are naturally interpreted in terms of, for example, insertion of complete sets of states; other formulae reveal some interesting and perhaps unsuspected properties of matrix elements involving what we term 'reservoir' states which are motivated naturally from issues of S-function generating functions. Finally we show how the S-function calculus may be used to compute the formula for the trace of a product of vertex operators. Although this can be computed by standard means [12, 18], the derivation in the S-function notation uses an elegant supersymmetric generalization of an S-function transcription of a MacDonald identity [16] and is worth presenting in its own right.

3.1. S-function generating functions and vertex-operator matrix elements

A natural concomitant of the theory of S-functions is the formal calculus of generating functions for certain infinite series (see [16] and also King [11]) which arise naturally in the application to finite-dimensional Lie algebras [9, 11] and which can be considerably generalized [19]. Here we consider their counterparts in the Fock space of the vertex operators, and their implications for matrix elements, products and traces of vertex operators. Generically such a generating function may have a particularly simple expression in terms of the indeterminates x_i; matrix elements of a product of vertex operators involving the corresponding Fock states must attain a similar form in the parameters z_i in terms of which the vertex operators are defined.

We note from (15) and the definition (12) of the composite S-function that for the special case of $\alpha_1 = \alpha_2 = \cdots = 1$ we have

$$(V_{\alpha_1} V_{\alpha_2} \cdots)_{\mu 0} = \prod_{i<j} \left(1 - \frac{z_j}{z_i}\right) s_\mu(z_1, z_2, \ldots). \tag{17}$$

The products in (17) can be formally treated as being infinite; and on the same footing as the S-function being a function of an infinite number of variables. In fact we can get rid of the product factor on the right-hand side of (17) by normal ordering the left-hand product. More specifically, we use the result (see, e.g., [15])

$$V_{\alpha_1}(z_1) \cdots V_{\alpha_n}(z_n) = \prod_{i<j} \left(1 - \frac{z_j}{z_i}\right)^{\alpha_i \alpha_j} : V_{\alpha_1}(z_1) \cdots V_{\alpha_n}(z_n) :$$

where by :: we mean the usual procedure of moving a_n to the right of a_m if $n > m$.

Generically a series defined by

$$Z_q = \sum_{\lambda \in Z} \zeta_\lambda q^{|\lambda|} s_\lambda(x) \tag{18}$$

will be associated with a 'reservoir' state of type Z_q,

$${}_q\langle Z| \equiv \sum_{\lambda \in Z} \langle \lambda | \zeta_\lambda q^{|\lambda|}$$

which is a formal coherent-like sum over all λ in the set Z, which may include all partitions or some distinguished series. Thus the overlap with the vacuum state of a product of vertex operators acting on such a reservoir state will regenerate the original generating relation, now in terms of the parameters z_i, and leads to some perhaps unsuspected identities for the vertex realization.

In appendix 2 are (see equations (5)) given generating functions for several of the standard infinite S-function series, and here we list the resulting vacuum–reservoir state vertex-operator matrix element following the above prescription:

$$
\begin{aligned}
{}_q\langle A| : V_1(z_1)V_1(z_2)\cdots : |0\rangle &= \prod_{i<j}(1-qz_iz_j)\\
{}_q\langle B| : V_1(z_1)V_1(z_2)\cdots : |0\rangle &= \prod_{i<j}(1-qz_iz_j)^{-1}\\
{}_q\langle C| : V_1(z_1)V_1(z_2)\cdots : |0\rangle &= \prod_{i\leqslant j}(1-qz_iz_j)\\
{}_q\langle D| : V_1(z_1)V_1(z_2)\cdots : |0\rangle &= \prod_{i\leqslant j}(1-qz_iz_j)^{-1}.
\end{aligned}
\qquad (19)
$$

It is convenient to define modified S-functions (and corresponding kets) $S_{[\lambda]}(x) \equiv S_{\lambda/C}(x)$, $\langle[\lambda]| \equiv \langle\lambda/C|$, $S_{<\lambda>}(x) \equiv S_{\lambda/A}(x)$, $\langle\langle\lambda\rangle| \equiv \langle\lambda/A|$, where the notation indicates S-function division distributed over all admissible elements of the indicated series. Then the MacDonald identities written in terms of S-functions in appendix 2 merely express matrix elements of vertex operators with the corresponding reservoir states:

$$
\begin{aligned}
{}_q\langle A/C| : V_1(z_1)V_1(z_2)\cdots : |0\rangle &= \prod_{k=1}^{\infty}\prod_{i<j}(1-q^kx_ix_j)\\
{}_q\langle C/A| : V_1(z_1)V_1(z_2)\cdots : |0\rangle &= \prod_{k=1}^{\infty}\prod_{i\leqslant j}(1-q^kx_ix_j).
\end{aligned}
\qquad (20)
$$

Further MacDonald identities in the S-function basis can be rewritten in an obvious way in terms of vertex-operator matrix elements, but the general transcription is obvious from these examples. In fact, the evaluation of the vertex-operator trace in the S-function basis necessitates a further MacDonald-type identity, this time for composite supersymmetric S-functions. This case is examined in the next subsection.

In closing, we would like to mention that the basic series $I_1(\bar{w};z)$ itself (cf (9)) has a natural interpretation in terms of the vacuum-to-vacuum matrix element of the normal ordered product of vertex operators evaluated at $w_1, w_2, \ldots$ with the normal ordered product of vertex operators evaluated at $z_1, z_2, \ldots$,

$$
I_1(\bar{w};z) = \prod_{i,j=1}^{\infty}\left(1-\frac{z_i}{w_j}\right) = \langle 0| : V_1(w_1)V_1(w_2)\cdots :: V_1(z_1)V_1(z_2)\cdots : |0\rangle.
$$

This follows from (17), (3.1) and the result

$$
\langle 0| : V_1(w_1)V_2(w_2)\cdots : |\nu\rangle = (-1)^{|\nu|}s_{\nu'}(\bar{w}_1, \bar{w}_2, \ldots).
$$

3.2. Vertex-operator trace formula and MacDonald identities

For the trace of an arbitrary product of vertex operators we need to sum the expression

$$\sum_{\mu}(-q)^{|\mu|}s_{\tilde{\mu};\mu'}((z_1,z_2,\dots,z_n)).$$

For the case when $n=1$ and $\alpha_1>0$, we note that the answer lies in the reformulation in the S-function language [16] of a MacDonald identity

$$\sum_{\mu}(-q)^{|\mu|}s_{\tilde{\mu};\mu'}(z)=\prod_{k=1}^{\infty}\left\{\frac{I_{q^k}(z;\bar{z})}{(1-q^k)}\right\} \tag{21}$$

where $I_q(x;y)$ is defined in (9). The supersymmetric generalization

$$\sum_{\mu}(-q)^{|\mu|}s_{\tilde{\mu};\mu'}((z_1,z_2,\dots,z_n))=\prod_{k=1}^{\infty}\frac{I_{q^k}((z_1,\dots,z_n;\bar{z}_1,\dots,\bar{z}_n))}{(1-q^k)} \tag{22}$$

which we shall need is derived in appendix 2, equation (53).

Recalling the result (15) for the matrix element of an arbitrary product of vertex operators,

$$\begin{aligned}(V_{\alpha_1}(z_1)\cdots V_{\alpha_n}(z_n))_{\mu\nu}&=\prod_{1\leqslant i<j\leqslant n}I_1((z_j;\bar{z}_i))(-1)^{|\nu|}s_{\tilde{\mu};\nu'}((z_1,z_2,\dots,z_n))\\&=\prod_{1\leqslant i<j\leqslant n}\left(1-\frac{z_j}{z_i}\right)^{\alpha_i\alpha_j}(-1)^{|\nu|}s_{\tilde{\mu};\nu'}((z_1,z_2,\dots,z_n))\end{aligned} \tag{23}$$

the trace of an arbitrary product of vertex operators can be evaluated as

$$\begin{aligned}&\mathrm{Tr}(V_{\alpha_1}(z_1)\cdots V_{\alpha_n}(z_n)q^{L_0})\\&\quad=\prod_{1\leqslant i<j\leqslant n}\left(1-\frac{z_j}{z_i}\right)^{\alpha_i\alpha_j}\sum_{\mu}(-q)^{|\mu|}s_{\tilde{\mu};\mu'}((z_1,z_2,\dots,z_n))\\&\quad=\prod_{1\leqslant i<j\leqslant n}\left(1-\frac{z_j}{z_i}\right)^{\alpha_i\alpha_j}\prod_{k=1}^{\infty}\frac{I_{q^k}((z_1,\dots,z_n;\bar{z}_1,\dots,\bar{z}_n))}{(1-q^k)}\\&\quad=\prod_{1\leqslant i<j\leqslant n}\left(1-\frac{z_j}{z_i}\right)^{\alpha_i\alpha_j}\frac{1}{\prod_{k=1}^{\infty}(1-q^k)}\prod_{k=1}^{\infty}\left[\prod_{i,j}\left(1-q^k\frac{z_i}{z_j}\right)^{\alpha_i\alpha_j}\right].\end{aligned} \tag{24}$$

Here, L_0 is the Virasoro operator defined by $L_0=\frac{1}{2}p_0^2+\sum_{n=1}^{\infty}a_{-n}a_n$. Using (22) we therefore have the result

$$\begin{aligned}\mathrm{Tr}(U_{\alpha_1}(z_1)\cdots U_{\alpha_n}(z_n)q^{L_0})&=\delta_{\sum\alpha_i,0}\prod_{i=1}^{n}z_i^{-\alpha_i^2/2}\prod_{1\leqslant i<j\leqslant n}\left(\frac{z_i-z_j}{z_iz_j}\right)^{\alpha_i\alpha_j}\\&\quad\times\frac{\sum_{\lambda\in\Gamma}q^{\lambda^2/2}(\prod_i z_i^{\alpha_i})^{\lambda}}{\prod_{k=1}^{\infty}(1-q^k)}\prod_{k=1}^{\infty}\left[\prod_{i,j}\left(1-q^k\frac{z_i}{z_j}\right)^{\alpha_i\alpha_j}\right].\end{aligned} \tag{25}$$

For the case of $A_1^{(1)}$, where the momentum lattice is $\Gamma=\mathbb{Z}\alpha$ and $\alpha=\sqrt{2}$ we have the well known result

$$\begin{aligned}\mathrm{Tr}(U_{\alpha}(z)U_{-\alpha}(w)q^{L_0})&=\frac{zw}{(z-w)^2}\frac{\sum_{m\in\mathbb{Z}}q^{m^2}(z/w)^{2m}}{\prod_{k=1}^{\infty}(1-q^k)}\\&\quad\times\prod_{k=1}^{\infty}\left[\frac{(1-q^k)^2}{(1-q^kz/w)(1-q^kw/z)}\right]^2.\end{aligned} \tag{26}$$

4. Vertex operators, S-functions and representations of affine algebras

In this section we consider the vertex operators (2) and (4) as a vehicle for construction of representations of various classes of infinite-dimensional algebra. Thus, for example, we have for $\alpha = 1$ a free fermion, $\alpha = \pm\sqrt{2}$ the currents for the non-zero roots of $A_1^{(1)}$, in the level-one vacuum representation, and for $\alpha = \pm\sqrt{3}$ the fermionic generators of a $c = 1$ realization of the $N = 2$ superconformal algebra [20]. Using the results of the previous sections, we first examine the explicit form of the relevant operator products as the points at which they are defined coalesce. This is done for arbitrary matrix elements $\langle p_\lambda, \lambda | U_\alpha(z) U_{-\alpha}(w) | p_\mu, \mu\rangle$ and a systematic expansion in powers of $(z - w)$ is shown to be constructible in principle. In particular we verify the singular terms in $\alpha = \pm\sqrt{2}$ case, and thereby the corresponding commutators of the associated infinite-dimensional Lie algebra.

In the case of finite-dimensional Lie algebras, an important tool [21] is the construction of an operator A in the tensor product of the enveloping algebra with the space of endomorphisms of a module termed the 'reference' representation. In terms of a basis $\{X_a\}$ of the Lie algebra and its dual basis $\{X^a\}$, we have $A = \sum_a X_a \otimes \pi(X^a)$, and A commutes with the diagonal action of the Lie algebra, so that characteristic identities and Casimir invariants can be defined in terms of it. The generalization of the former has been exhibited for the Kac–Moody case [22]; the latter requires suitably defined traces over the 'reference' representation, and the present S-function basis suggests itself as an appropriate means of handling the construction in this case.

Taking up the $A_1^{(1)}$ case, a natural current operator $A(z)$ is introduced on the tensor product of an arbitrary representation with the vertex (level-one vacuum) representation as 'reference' representation, which is the analogue of the operator A in the finite-dimensional case. We shall see that, in an appropriate contour integral, the regulated partial trace of $A(z)A(w)$ over the 'reference' representation gives essentially the energy–momentum tensor, thus recovering the Sugawara construction from a trace formula in this framework.

4.1. *Operator product algebra in the S-function basis*

We shall make use of the general expressions for matrix elements developed in the S-function basis in section 2, for the case $\langle \lambda, p_\lambda | U_\alpha(z) U_{-\alpha}(w) | \mu, p_\mu\rangle$. In the $A_1^{(1)}$ case we have $\alpha = \sqrt{2}$, and the vertex operators in the standard way [3] become the currents for the positive and negative roots $\pm\alpha$ up to cocycle factors. Here and below we denote $U_{\pm\sqrt{2}}(z) \equiv z j_\pm(z)$, and the current corresponding to the Cartan element is $j_0(z) = h(z)/\sqrt{2}$, where

$$h(z) = p_0 + \sum_{n=1}^{\infty} (a_n z^{-n} + a_{-n} z^n). \tag{27}$$

From the general form (15) and (14) we have on writing $z = w + (z - w)$ in the prefactor $I_1(\bar{z}/0; 0/\bar{w})$ the result

$$\begin{aligned}\langle \lambda, p_\lambda | U_\alpha(z) U_{-\alpha}(w) | \mu, p_\mu\rangle &\\ = \delta_{p_\lambda p_\mu} \left(\frac{z-w}{w}\right)^{-\alpha^2} &\left\{1 + \frac{z-w}{w}\alpha(p_\lambda + \tfrac{1}{2}\alpha) + \mathrm{O}\left(\frac{z-w}{w}\right)^2\right\} \\ &\times (-1)^{|\mu|} s_{\bar{\lambda};\mu'}(\bar{z}/\bar{w})\end{aligned} \tag{28}$$

where the limit $w \to z$ still has to be taken in the S-function. For the latter task we recall from section 2, the generating function

$$\sum_{\mu,\nu} s_\mu(r)s_\nu(t)(-1)^{|\nu|}s_{\tilde{\mu};\nu'}(\tilde{z},0/0,\tilde{w}) = J_1(r;t)J_1(r;z,0/0,w)I_1(t;\tilde{z},0/0,\tilde{w}). \tag{29}$$

Consider, for example, the product $J_1(r;z,0/0,w)$:

$$\left(\frac{\prod_i(1-r_iz)}{\prod_i(1-r_iw)}\right)^{-\alpha} = \left(\prod_i\left(1-\frac{r_i(z-w)}{(1-r_iw)}\right)\right)^{-\alpha} = \left(\prod_i\left(1-\frac{(z-w)}{w}\tilde{r}_i\right)\right)^{-\alpha}$$

$$= \sum_\rho s_\rho\left(\frac{z-w}{w}\right)s_\rho(\tilde{r}) = \sum_\rho \binom{\alpha}{\rho'}\left(\frac{z-w}{w}\right)^{|\rho|}s_\rho(\tilde{r}) \tag{30}$$

where

$$\tilde{r}_i = \frac{r_iw}{(1-r_iw)} = r_iw + (r_iw)^2 + (r_iw)^3 + \cdots. \tag{31}$$

Now symmetric functions of the $\tilde{r}_i$ may be re-expressed in terms of the r_i using the notion of plethysm, $\alpha \otimes \beta$ [8]. In view of (31) and recalling (7) in section 2, we have

$$\begin{aligned} s_{(1)}(\tilde{r}) &= \sum_n w^n p_n(r) = \sum_{n\geqslant 1}\sum_{|\rho|=n} K^{-1}{}_{(n)\rho}w^n s_\rho(r) \\ &= ws_{(1)}(r) + w^2s_{(2)}(r) - w^2s_{(11)}(r) + \cdots \end{aligned} \tag{32}$$

so that

$$s_\lambda(\tilde{r}) = s_{((1)+(2)-(11)+\cdots)\otimes\lambda}(r). \tag{33}$$

In the present case ($\alpha = \sqrt{2}$) the general expression for the distributive law for plethysms [9] is not required, and from (30) we have

$$J_1(r;z,0/0,w) = 1 + \alpha\left(\frac{z-w}{w}\right)\sum_{n\geqslant 1,|\rho|=n} K^{-1}{}_{(n)\rho}w^n s_\rho(r) + \mathrm{O}\left(\frac{z-w}{w}\right)^2 \tag{34}$$

and the same reasoning leads to

$$I_1(t;\tilde{z},0/0,\tilde{w}) = 1 - \alpha\left(\frac{z-w}{z}\right)\sum_{m\geqslant 1,|\sigma|=m} K^{-1}{}_{(m)\sigma}w^{-m}s_\sigma(t) + \mathrm{O}\left(\frac{z-w}{z}\right)^2 \tag{35}$$

involving S-functions of t_i. The final result is obtained when the last two equations are substituted into (28) leading to the generating function in the form

$$\begin{aligned} &\sum_{\lambda,\mu} s_\lambda(r)\langle\lambda,p_\lambda|\frac{1}{z}U_\alpha(z)\frac{1}{w}U_{-\alpha}(w)|\mu,p_\mu\rangle s_\mu(t)(-1)^{|\mu|} \\ &\quad = \delta_{p_\lambda p_\mu}(z-w)^{-\alpha^2}\Bigg(\sum_\lambda s_\lambda(r)s_\lambda(t) + \alpha\frac{(z-w)}{z}\Bigg(p_\lambda\sum_\lambda s_\lambda(r)s_\lambda(t) \\ &\qquad + \sum_\lambda\sum_{m\geqslant 1,|\sigma|=m} K^{-1}{}_{(m)\sigma}w^{-m}s_\lambda(r)s_{\sigma.\lambda}(t) \\ &\qquad + \sum_\mu\sum_{n\geqslant 1,|\rho|=n} K^{-1}{}_{(n)\rho}w^n s_{\rho.\mu}(r)s_\mu(t)\Bigg)\Bigg) \end{aligned}$$

up to $O(z-w)^2$. Finally this result must be compared with matrix elements of $h(z)$. From (27) above and (5) in section 2 we have the generating function

$$\sum_{\lambda,\mu} s_\lambda(r)\langle\lambda,p_\lambda|h(z)|\mu,p_\mu\rangle s_\mu(t)$$

$$= (\delta_{p_\lambda p_\mu})\Big(p_\lambda \sum_\lambda s_\lambda(r)s_\lambda(t) + \sum_{\lambda,\mu}\sum_{n=1}^{\infty} z^{-n} s_\lambda(r)\langle p_n(x)s_\lambda(x), s_\mu(x)\rangle$$

$$\times\, s_\mu(t) + \sum_{\lambda,\mu}\sum_{n=1}^{\infty} z^{n} s_\lambda(r)\langle s_\lambda(x), p_n(x)s_\mu(x)\rangle s_\mu(t)\Big)$$

$$= (\delta_{p_\lambda p_\mu})\Big(\sum_\lambda p_\lambda s_\lambda(r)s_\lambda(t) + \sum_\lambda \sum_{m\geqslant 1, |\sigma|=m} K^{-1}{}_{(m)\sigma} z^{-m} s_\lambda(r) s_{\sigma.\lambda}(t)$$

$$+ \sum_\mu \sum_{n\geqslant 1,|\rho|=n} z^n K^{-1}{}_{(n)\rho} z^n s_{\rho.\mu}(r) s_\mu(t)\Big) \tag{36}$$

where (7) and (2) and the properties of the inner product have been used. Comparison of (35) and (36) establish, at the level of *arbitrary* matrix elements, the operator product expansion valid for the $A_1{}^{(1)}$ case,

$$j_+(z)j_-(w) = (z-w)^{-2} + 2(z-w)^{-1}\sqrt{2}j_0(w)$$

with central charge $k=1$ as required.

4.2. The Sugawara construction and S-function traces

Corresponding to the currents $j_\pm(z), j_0(z)$ providing the level-one vacuum representation of $A_1{}^{(1)}$, we introduce the currents $J_\pm(z), J(z)$ which act in an arbitrary representation (or, more generally, in the enveloping algebra of $A_1{}^{(1)}$). Consider the combination

$$A(z) = J_0(z)\otimes j_0(z) + J_+(z)\otimes j_-(z) + J_-(z)\otimes j_+(z). \tag{37}$$

Now

$$\oint zA(z)\,\mathrm{d}z = \sum_{i=1}^{3}\sum_{n\in\mathbb{Z}} J^i(n)\otimes j_i(-n)$$

where $\{J^i\},\{J_i\}$ are dual bases for the Lie algebra of A_1, $[J_i,J_j] = \epsilon_{ijk}J_k$, and $\{j^i(z)\},\{j_i(z)\}$ the corresponding currents in the vertex representation. Hence $A(z)$ in (37) is the natural generalization of the central invariant A which can be used to formulate characteristic identities for both the Lie algebra [21] and Kac–Moody [22] cases. Henceforth we drop the $\otimes$ for notational simplicity.

In the light of the above remarks we introduce the regulated trace of the product

$$4\mathrm{Tr}_q(A(z)A(w)) = \mathrm{Tr}\{(H(z)h(z) + J_+(z)j_-(z) + J_-(z)j_+(z))$$

$$\times\,(H(w)h(w) + J_+(w)j_-(w) + J_-(w)j_+(w))q^d\}$$

$$= H(z)H(w)\,\mathrm{Tr}_q(h(z)h(w)) + J_+(z)J_-(w)\,\mathrm{Tr}_q(j_-(z)j_+(w))$$

$$+ J_-(z)J_+(w)\,\mathrm{Tr}_q(j_+(z)j_-(w)) \tag{38}$$

where we have used momentum conservation to select only the diagonal combinations in momentum space and $d \equiv L_0$.

The traces can be evaluated using the S-function basis and the results of section 3. In order to ensure convergence in the present case of infinite-dimensional algebras, *normal ordering must be ensured—in the finite-dimensional case this step is not* necessary. At the level of the operator $A(z)$ this is incorporated via radial ordering in the contour integral [23],

$$\begin{aligned}(AA)_q(w) &\equiv \oint_w \mathrm{d}z\,(z-w)\,\mathrm{Tr}_q(\mathbb{R}A(z)A(w)) \\ &= \oint_> \mathrm{d}z\,(z-w)\,\mathrm{Tr}_q(A(z)A(w)) - \oint_< \mathrm{d}z\,(z-w)\,\mathrm{Tr}_q(A(w)A(z)).\end{aligned} \tag{39}$$

Each term of (39) is of the form (38), and the labels $<$ and $>$ stand for the contours $|z| < |w|$ and $|z| > |w|$ respectively. We consider the contributions in pairs where the same operators are involved but with different ordering. Consider, for example,

$$\begin{aligned}&\oint_> \mathrm{d}z\,(z-w)J_-(z)J_+(w)\,\mathrm{Tr}(j_+(z)j_-(w)q^d) \\ &\qquad - \oint_< \mathrm{d}z\,(z-w)J_+(w)J_-(z)\,\mathrm{Tr}(j_-(w)j_+(z)q^d) \\ &\quad = \oint_> \mathrm{d}z\,J_-(z)J_+(w)\frac{zw}{z-w}\left(\frac{\sum_n q^{n^2}(z/w)^{2n}}{\prod_n(1-q^n)}\right) \\ &\qquad \times \prod_{k=1}^{\infty}\left(\frac{1-q^k}{1-q^k z/w}\right)^2\left(\frac{1-q^k}{1-q^k w/z}\right)^2 \\ &\qquad - \oint_< \mathrm{d}z\,J_+(w)J_-(z)\frac{zw}{z-w}\left(\frac{\sum_n q^{n^2}(z/w)^{2n}}{\prod_n(1-q^n)}\right) \\ &\qquad \times \prod_{k=1}^{\infty}\left(\frac{1-q^k}{1-q^k z/w}\right)^2\left(\frac{1-q^k}{1-q^k w/z}\right)^2\end{aligned} \tag{40}$$

where the result (26) has been used for $\alpha = \sqrt{2}$. Examination of the terms in the infinite products shows that singularities appear at $z = wq^{\pm k}$ in the $|z| < |w|$, $|z| > |w|$ cases respectively. Thus for fixed $|q| < 1$, a contour can be chosen to avoid these double poles, and the remaining contours combined into a contour around the simple pole $z = w$, if the radial ordering convention is understood. Moreover, since the contour will pick out the residue of the integrand at the pole, the substitution $z = w$ may be made in the q-dependent part.

The same result can obviously be obtained for the $J_+(z)J_-(w)$ contribution to (39). The contribution from the $H(z)H(w)$ terms is more difficult but can be derived from the vertex-operator trace by suitable differentiation. First we write

$$\mathrm{Tr}\,h(z)h(w)q^d = \mathrm{Tr}((h(z)-p_0)(h(w)-p_0)q^d) + \mathrm{Tr}(p_0)^2q^d$$

where the off-diagonal nature of $(h(z)-p_0)$ has been used. Next since

$$z\frac{\mathrm{d}}{\mathrm{d}z}\left(\left.\frac{\partial V}{\partial\alpha}\right|_{\alpha=0}\right)=h(z)-p_0$$

we have

$$\operatorname{Tr}h(z)h(w)q^d=z\frac{\mathrm{d}}{\mathrm{d}z}w\frac{\mathrm{d}}{\mathrm{d}w}\left[-\frac{\partial}{\partial\alpha}\frac{\partial}{\partial\beta}\operatorname{Tr}(V_{+\alpha}(z)V_{-\beta}(w)q^d)\right]_{\alpha=\beta=0}+\operatorname{Tr}((p_0)^2q^d). \tag{41}$$

Including the trace over momentum states, the result of the differentiations gives a formula comparable to (26),

$$\begin{aligned}\operatorname{Tr}h(z)h(w)q^d&=\left(\frac{\sum_n q^{n^2}(z/w)^{2n}}{\prod_n(1-q^n)}\right)\\&\quad\times\left(\frac{zw}{(z-w)^2}+\sum_k\frac{(q^{2k}+1)(q^kw/z+q^kz/w)-4q^{2k}}{(1-q^kw/z)(1-q^kz/w)}\right)\\&\quad+\left(\frac{\sum_n n^2q^{n^2}}{\prod_n(1-q^n)}\right).\end{aligned}\tag{42}$$

This result can now be used in an analogous manner to (26) to establish the $H(z)H(w)$ contribution to (39):

$$\begin{aligned}&\oint_> \mathrm{d}z\,(z-w)H(z)H(w)\operatorname{Tr}(h(z)h(w)q^d)\\&\qquad-\oint_< \mathrm{d}z\,(z-w)H(w)H(z)\operatorname{Tr}(h(w)h(z)q^d)\\&\quad=\oint_> \mathrm{d}z\,H(z)H(w)\frac{zw}{z-w}\left(\frac{\sum_n q^{n^2}}{\prod_n(1-q^n)}\right)\\&\qquad\times\left(1+\frac{(z-w)^2}{zw}\sum_k\frac{(q^{2k}+1)(q^kw/z+q^kz/w)-4q^{2k}}{(1-q^kw/z)(1-q^kz/w)}\right)\\&\qquad-\oint_< \mathrm{d}z\,H(w)H(z)\frac{zw}{z-w}\left(\frac{\sum_n q^{n^2}}{\prod_n(1-q^n)}\right)\\&\qquad\times\left(1+\frac{(z-w)^2}{zw}\sum_k\frac{(q^{2k}+1)(q^kw/z+q^kz/w)-4q^{2k}}{(1-q^kw/z)(1-q^kz/w)}\right)\\&\qquad+\left(\frac{\sum_n n^2q^{n^2}}{\prod_n(1-q^n)}\right)\left(\oint_> \mathrm{d}z\,(z-w)H(z)H(w)\right.\\&\qquad\left.-\oint_< \mathrm{d}z\,(z-w)H(w)H(z)\right).\end{aligned}\tag{43}$$

In this case, because of the persistence of $(z-w)$ factors, only the leading terms in the first two integrals have simple poles at $z=w$. The last two integrals have *no* simple poles, while the single poles at $z = wq^{\pm k}$ in the $|z| < |w|$, $|z||w|$ cases which now appear (cf (40)) can again be avoided for fixed q by suitably chosen contours.

Combining (40) with (43), we have simply

$$4(AA)_q(w) = \left(\frac{\sum_n q^{n^2}}{\prod_n(1-q^n)}\right) \times \oint_w \mathrm{d}z\, \frac{H(z)H(w) + (J_+(z)J_-(w) + J_-(z)J_+(w))}{(z-w)}. \tag{44}$$

Including the appropriate normalization factor to remove the singular q-dimension, the limit $q \to 1$ can be taken and the result is essentially the current corresponding to the Virasoro algebra:

$$L(w) = \lim_{q\to 1}\left(\frac{\prod_n(1-q^n)}{\sum_n q^{n^2}}\right)(AA)_q(w). \tag{45}$$

As well as being an application of the trace formulae for vertex-operator products, this result shows that the Sugawara construction [24] can be recovered entirely in terms of the operator $A(z)$ and demonstrates the utility of this construction. The relationship of $A(z)$ to the characteristic identity [22] and other properties can be expected to be established with the help of the S-function realization of the reference representation furnished by the vertex operators [25].

5. Conclusions

The goal of the present paper has been to point out the intimate connection between vertex-operator constructions and the theory of S-functions: as we have shown, vertex-operator matrix elements are basically composite or universal S-functions of their arguments. As a verification of the techniques, we have derived certain trace, operator product and current algebra constructions in the S-function basis. On the other hand, it turns out that certain combinatorial identities and MacDonald identities involving S-functions have an extremely natural transcription as matrix elements in Fock space.

Further development of this work can be expected through the full exploitation of the theory of S-functions to vertex-operator constructions. For example, vertex operators for root lattices of rank greater than one should be associated with S-functions of compound arguments; nor have the implications of the Littlewood–Richardson rule for manipulation of vertex-operator matrix elements been fully examined. Also, the use of non-orthonormal bases such as the kets $|[\lambda]\rangle$, $|\langle\lambda\rangle\rangle$ which arise naturally in connection with MacDonald identities, together with their dual bases $|[\lambda]^*\rangle \equiv |\lambda \cdot D\rangle_q$, $|\langle\lambda\rangle^*\rangle \equiv |\lambda \cdot B\rangle_q$, (defined for $|q| < 1$), may have interesting applications for realizations of affine algebras.

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Appendix 1. Some useful formulae

We summarize here some useful formulae used throughout the text.

$$s_\lambda(x/y) \equiv \sum_\nu (-1)^{|\nu|} s_{\lambda/\nu}(x) s_{\nu'}(y) = (-1)^{|\lambda|} s_{\lambda'}(y/x)$$

$$s_{\bar\nu;\mu}(x/y) \equiv \sum_\zeta s_{\nu/\zeta}(\bar x/\bar y) s_{\mu/\zeta'}(x/y) \qquad s_\alpha(x,r/y) = \sum_\beta s_{\alpha/\beta}(x/y) s_\beta(r)$$

$$s_\alpha(x/y,s) = \sum_\beta (-1)^{|\beta|} s_{\alpha/\beta}(x/y) s_{\beta'}(s) \qquad s_\lambda(x,y) = \sum_\nu s_{\lambda/\nu}(x) s_\nu(y)$$

$$I_q(x;y) = \sum_\alpha s_\alpha(x) s_{\alpha'}(y)(-q)^{|\alpha|} = \prod_{i,j}(1-qx_iy_j)$$

$$J_q(x;y) = \sum_\alpha s_\alpha(x) s_\alpha(y) = \prod_{i,j}(1-qx_iy_j)^{-1}$$

$$I_q(x/y;t) = \sum_\alpha s_\alpha(x/y) s_{\alpha'}(t)(-q)^{|\alpha|} = I_q(x;t) J_q(y;t)$$

$$J_q(x/y;t) = \sum_\alpha s_\alpha(x/y) s_\alpha(t) = J_q(x;t) I_q(y;t)$$

$$I_q(x/y;s/t) = \sum_\alpha s_\alpha(x/y) s_{\alpha'}(s/t)(-q)^{|\alpha|} = I_q(x;s) I_q(y;t) J_q(x;t) J_q(y;s)$$

$$J_q(x/y;s/t) = \sum_\alpha s_\alpha(x/y) s_\alpha(s/t) = J_q(x;s) J_q(y;t) I_q(y;s) I_q(x;t). \qquad \text{(A1.1)}$$

Appendix 2. S-function series

We give the notation for several of the standard infinite S-function series frequently used in the representation theory of finite-dimensional algebras and superalgebras [16], and which are given a Fock space interpretation in section 3 (see equations (19)):

$$A_q(x) = \prod_{i<j}(1-qx_ix_j) = \sum_{\alpha\in A} (-q)^{|\alpha|/2} s_\alpha(x)$$

$$B_q(x) = \prod_{i<j}(1-qx_ix_j)^{-1} = \sum_{\beta\in B} q^{|\beta|/2} s_\beta(x)$$

$$C_q(x) = \prod_{i\leqslant j}(1-qx_ix_j) = \sum_{\gamma\in C} (-q)^{|\gamma|/2} s_\gamma(x)$$

$$D_q(x) = \prod_{i\leqslant j}(1-qx_ix_j)^{-1} = \sum_{\delta\in D} (-q)^{|\delta|/2} s_\delta(x)$$

where, in Frobenius notation, A and C are sets of partitions of the form

$$\begin{pmatrix} a_1 & a_2 & \cdots \\ a_1+1 & a_2+1 & \end{pmatrix}, \begin{pmatrix} a_1+1 & a_2+1 & \cdots \\ a_1 & a_2 & \end{pmatrix}$$

D is the set of partitions all of whose parts are even, and B is the set of partitions all of whose distinct parts are repeated an even number of times.

MacDonald identities arise in this notation when infinite products are taken [16], and these also have their interpretations in Fock space, as shown in section 3 , for example for the following two series:

$$\prod_{k=1}^{\infty} A_{q^k}(x)_N = \sum_{\alpha \in A} (-q)^{|\alpha|/2} s_{[\alpha]}(x)_N \tag{A2.1}$$

$$\prod_{k=1}^{\infty} C_{q^k}(x)_N = \sum_{\gamma \in C} (-q)^{|\gamma|/2} s_{<\gamma>}(x)_N . \tag{A2.2}$$

Here the modified S-functions have been introduced in section 3 (see also conclusions).

Appendix 3. A MacDonald identity

In this appendix, we present a calculation crucial to the evaluation of the trace of vertex operators. We first define

$$S_{I_p/I_q}(x/y; r/t) \equiv \sum_{\mu,\nu} (-p)^{|\mu|}(-q)^{|\nu|} s_{\mu/\nu}(x/y) s_{\mu'/\nu'}(r/t) \tag{A3.1}$$

and

$$A_{\mu,\lambda}(x/y; r/t) \equiv \sum_{\zeta} (-p)^{|\zeta|} s_{\zeta/\mu}(x/y) s_{\zeta'/\lambda'}(r/t). \tag{A3.2}$$

We treat $A_{\mu,\lambda}$ as we did $B_{\mu,\lambda}$ in section 2. We sum the series $\sum_{\mu\lambda} A_{\mu,\lambda}(x/y; r/t) s_\mu(u) s_\lambda(v)$, rewrite it as another series and compare coefficients of $s_\mu(u)s_\lambda(v)$ in the two expressions. We find that

$$\begin{aligned}\sum_{\mu\lambda} A_{\mu,\lambda}(x/y; r/t) s_\mu(u) s_\lambda(v) &= \sum_{\zeta} (-p)^{|\zeta|} s_\zeta(x, u/y) s_{\zeta'}(r, v/t) \\ &= I_p(x, u/y; r, v/t) \\ &= I_p(x/y; r/t) I_p(x/y; v) I_p(r/t; u) I_p(u; v)\end{aligned}$$

where we have freely used results in appendix 1. The last three products can be rewritten as

$$\begin{aligned}I_p(x/y; v) I_p(r/t; u) I_p(u; v) &\\ &= \sum_{\zeta,\lambda,\beta} (-p)^{|\zeta|+|\lambda|+|\beta|} s_\zeta(x/y) s_{\zeta'}(v) s_\lambda(r/t) s_{\lambda'}(u) s_\beta(u) s_{\beta'}(v) \\ &= \sum_{\mu,\lambda,\rho} (-p)^{|\mu|+|\lambda|-|\rho|} s_{\lambda'/\rho}(x/y) s_{\mu'/\rho'}(r/t) s_\lambda(v) s_\mu(u).\end{aligned}$$

Therefore, we have

$$A_{\mu,\lambda}(x/y; r/t) = I_p(x/y; r/t) \sum_{\rho} (-p)^{|\mu|+|\lambda|-|\rho|} s_{\lambda'/\rho}(x/y) s_{\mu'/\rho'}(r/t). \tag{A3.3}$$

Hence we have the important identity

$$S_{I_p/I_q}(x/y;r/t) = \sum_{\mu}(-q)^{|\mu|}A_{\mu,\mu'}(x/y;r/t)$$
$$= I_p(x/y;r/t)S_{I_{qp^2}/I_{p-1}}(x/y;r/t). \tag{A3.4}$$

In the non-supersymmetric situation, i.e. when $y = t = 0$, this identity was crucial in King's reformulation of a MacDonald identity. In fact, what is required for evaluating the trace of an arbitrary product of vertex operators is the supersymmetric generalization of this MacDonald identity! Specifically, the identity is

$$S_{I_q/I}(x/y;r/t) = \prod_{k=1}^{\infty}\left\{\frac{I_{q^k}(x/y;r/t)}{(1-q^k)}\right\}. \tag{A3.5}$$

To prove this, we iterate (52) to obtain

$$S_{I_q/I}(x/y;r/t) = \prod_{k=1}^{n} I_{q^k}(x/y;r/t)S_{I_{q^{n+1}}/I_{q^{-n}}}(x/y;r/t)$$

for any n. Now

$$S_{I_{q^{n+1}}/I_{q^{-n}}}(x/y;r/t) = \sum_{\mu,\nu}(-q^{n+1})^{|\mu|}(-q^{-n})^{|\nu|}s_{\mu/\nu}(x/y)s_{\mu'/\nu'}(r/t)$$
$$= \sum_{\zeta} q^{|\zeta|} + \sum_{|\mu|>|\nu|}(-q)^{|\mu|}q^{n(|\mu|-|\nu|)}(-1)^{|\nu|}s_{\mu/\nu}(x/y)s_{\mu'/\nu'}(r/t)$$
$$\to \sum_{\zeta} q^{|\zeta|}$$

as $n \to \infty$ for small enough $|q|$. Now $\sum_{\zeta} q^{|\zeta|} = \prod_{k=1}^{\infty}(1-q^k)^{-1}$ and thus identity (A3.5) follows.

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